Average case complexity and the n-body problem [1]

Akitoshi Kawamura, Holger Thies, Martin Ziegler

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[1] The authors thank the Japan Society for the Promotion of Science (JSPS), Core-to-Core Program (A. Advanced Research Networks) and JSPS Kakenhi Project 2670000 for supporting the research.

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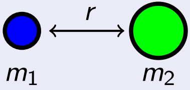

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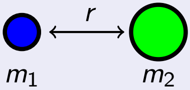

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Equations of Motion

$3N$-dim. 2nd order ODE system:

$$m_i \ddot{q}_i(t) = \sum_{k \neq i} \frac{m_i m_k (q_k(t) - q_i(t))}{|q_k(t) - q_i(t)|^3}$$

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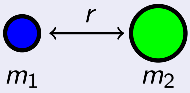

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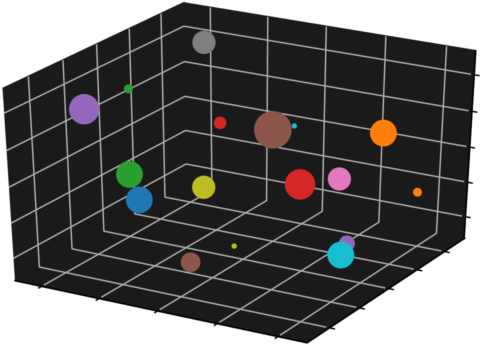
The system can equivalently be written as a $6N$-dimensional system of first-order ordinary differential equations.

N-body simulation



Problem

Given initial values $q_1, \ldots, q_N, v_1, \ldots, v_N$ and time t compute $q(t), v(t)$.

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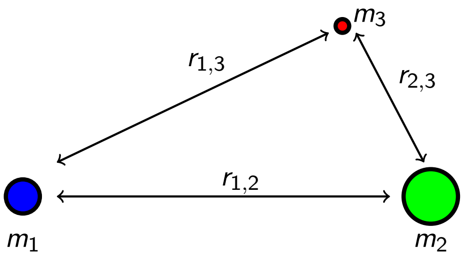
- However, the solution is not useful for computations as it converges extremely slowly.

Local solution

$$\ddot{q}_i(t) = \sum_{k \neq i} \frac{m_k(q_k(t) - q_i(t))}{\|q_k(t) - q_i(t)\|^3}$$

$$M := \sum_{i=1}^{N} m_i$$



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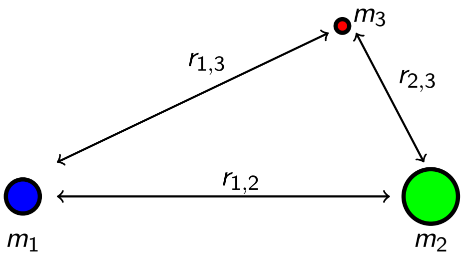

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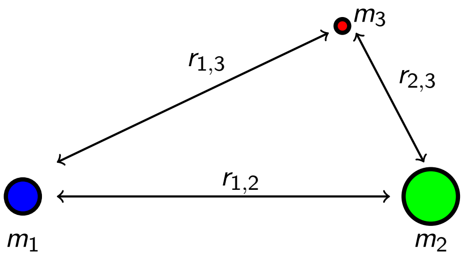

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- If $\|q_i(t) - q_i(t_0)\| \leq \frac{r}{4}$ it holds $r_{i,j}(t) \geq \frac{r}{2}$ and $\|\ddot{q}_i(t)\| \leq 4Mr^{-2}$

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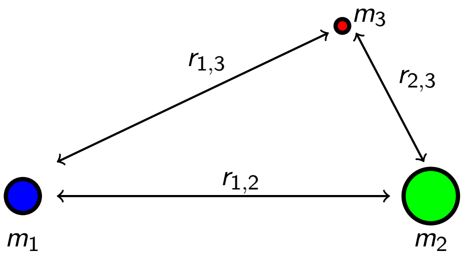

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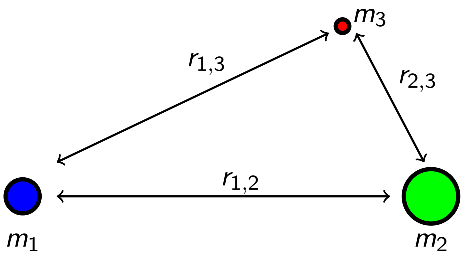

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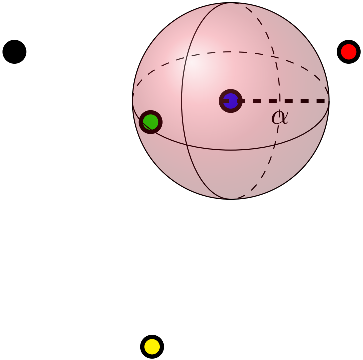
- Cauchy's existence theorem: $|t - t_0| < \frac{r^3}{(N+1)16M}$.
- For $|t - t_0| \leq \frac{r'}{2}$ it is suffices to sum $O(n)$ coefficients.



Assume $q(t), v(t) \notin N(\alpha)$ for $t \in [0, 1]$ then $q(t)$ can be computed in time $poly(n + 1/\alpha)$.

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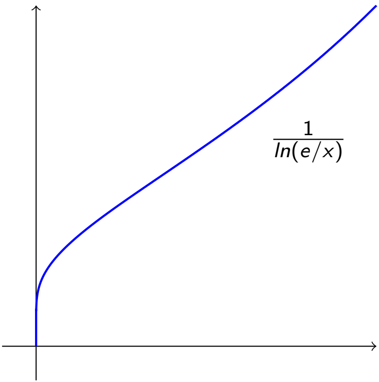
- Not robust w.r.t to polynomial slowdown
- More robust definition (Levin): $E_{P_n}\{\frac{T_{\mathcal{A}}(x)^{\varepsilon}}{n} \,:\, |x| \leq n\}$ bounded.
- Polynomial time on average: Probability of time longer than T is less than $\frac{poly(n)}{T^{\varepsilon}}$.

Average case complexity of real functions



Definition (Average Case Polynomial Time)

$T_A(x,n) := \max\{\, T_A((a_m)_{m\in\mathbb{N}}, n)) \ : \ a_m$ converges quickly to x$\}$
Polynomial average time: $\int \frac{T_A(x,n)^\varepsilon}{n} dx$ bounded.

Average case complexity for the N-body problem

- Recall α-collision: Two particles are α-close to each other
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The set $B(\alpha) \subseteq \mathbb{R}^{6n}$ is defined as the set of points (q_0, v_0) such that

1. (q_0, v_0) is an initial condition at time $t = 0$
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- Saari: The set of initial values leading to collisions for $N \leq 4$ has measure 0.

Hamiltonian systems

Definition

A Hamiltonian system is a dynamical system where the evolution over time is described by $2n$ first order ordinary differential equations of the form

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}$$

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Theorem (Liouville)

Let φ be a Hamiltonian system and A a set of inital conditions then

$$\int_A dz = \int_{\varphi_t(A)} dz.$$

The N-body problem in Hamiltonian form

The Hamiltonian for the N-body problem is

$$H(q,p) = \sum_{i=1}^{n} \frac{\|p_i\|^2}{2m_i} - \sum_{1 \leq i \leq j} \frac{m_i m_j}{\|q_i - q_j\|}$$

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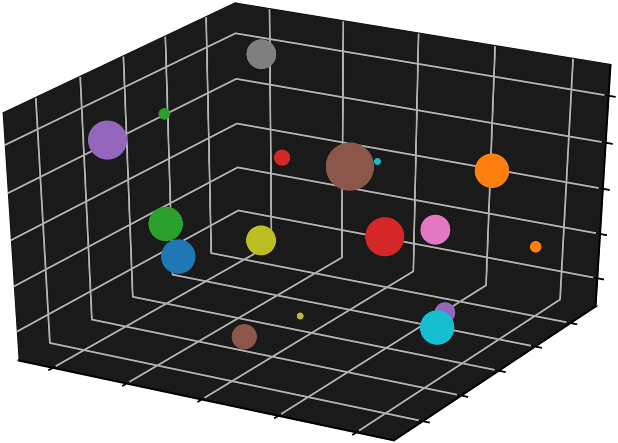
Basic Idea

- Show that the subset of phase space where two particles are close to each other is small.
- Apply Lioville's theorem and show that the corresponding set of initial values is small.

Restricting the problem



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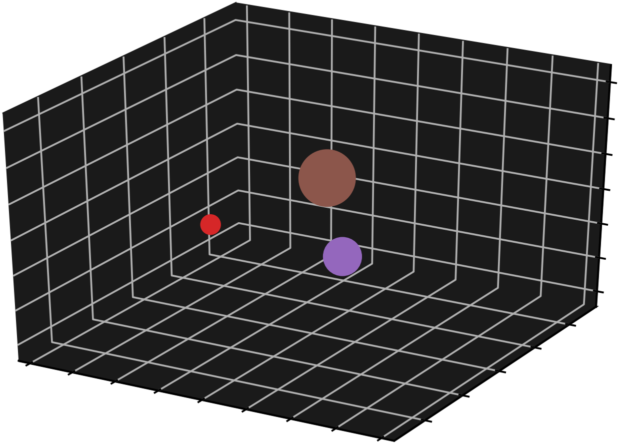
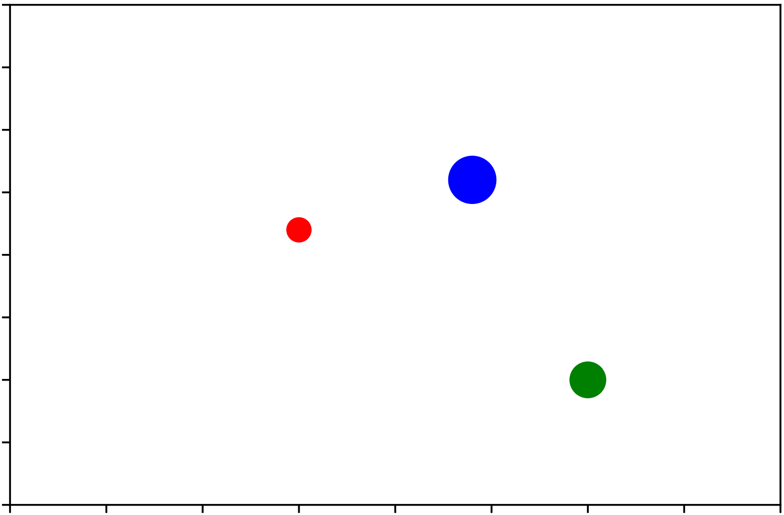


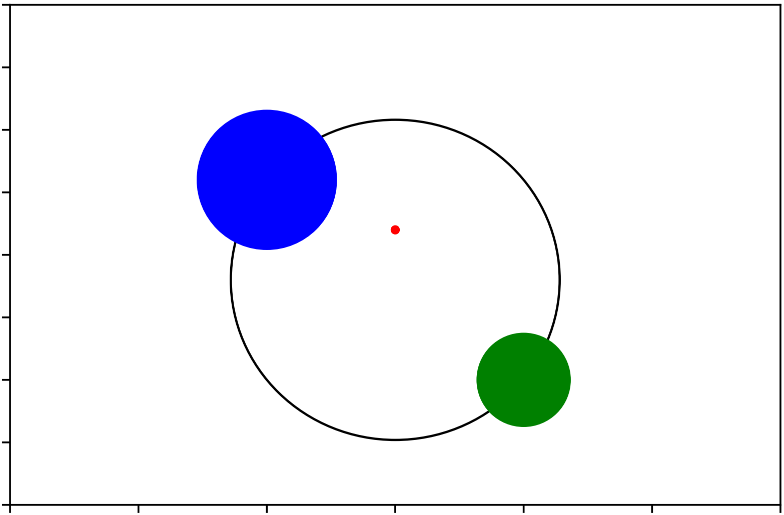
Three-body problem

Restricting the problem



Planar Three-body problem

Restricting the problem

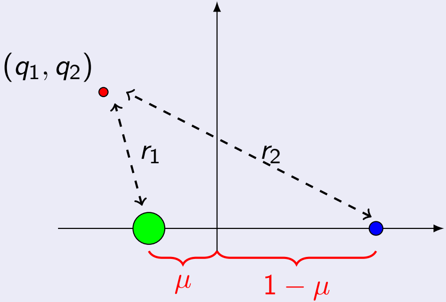


Planar Circular Restricted Three-body problem

The planar circular restricted three-body problem

Normalization

- $\mu \in [0, 0.5]$
- $m_1 = 1 - \mu$
- $m_2 = \mu$
- Position of m_1: $(-\mu, 0)$
- Position of m_2: $(1 - \mu, 0)$
- $r_1^2 = (q_1 + \mu)^2 + q_2^2$
- $r_2^2 = (q_1 - 1 + \mu)^2 + q_2^2$



The planar circular restricted three-body problem

Hamiltonian

The Hamiltonian of the planar restricted three body problem is

$$H(p, q) = \frac{1}{2}\|p\|^2 + q_2 p_1 - q_1 p_2 - \frac{\mu}{r_1} - \frac{1-\mu}{r_2}.$$

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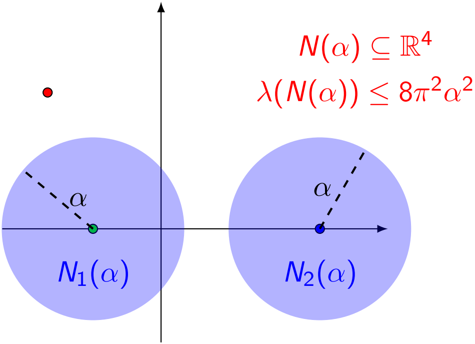
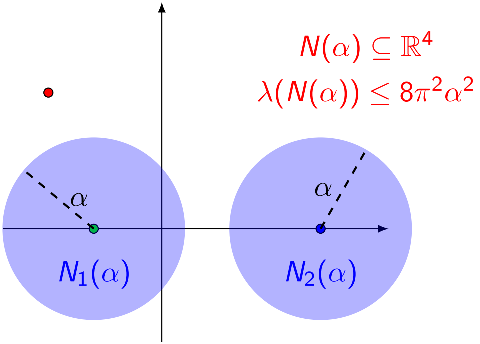
Planar restricted three body simulation

- $\mu \in [0, 0.5]$ fixed
- $A \subseteq \mathbb{R}^4$ the set of initial values (p, q) such that $H(p, q) \leq 1$ and $\|q\| \leq 1$.
- Goal: map $(p, q) \in A$ and $t \in [0, 1]$ to $q(t)$.


$N(\alpha) \subseteq \mathbb{R}^4$
$\lambda(N(\alpha)) \leq 8\pi^2\alpha^2$
α
α
$N_1(\alpha)$
$N_2(\alpha)$

α-collisions



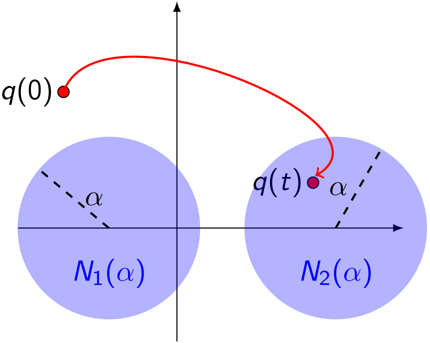
Proof Sketch.

- Change coordinates such that $(-\mu, 0)$ is at the origin
- Parameterize phase space by $\Phi : (H, r, \varphi, \psi)$
- $N_1(\varepsilon) \subseteq \Phi(G)$ for $G := [-1, 1] \times [0, \alpha] \times [0, 2\pi) \times [0, 2\pi)$

$\square$

Retracting to Initial Values

- $B_t(\alpha)$: Initial conditions ending up in $N(\alpha)$ at time t.



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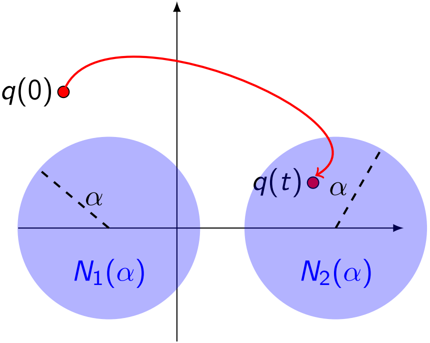

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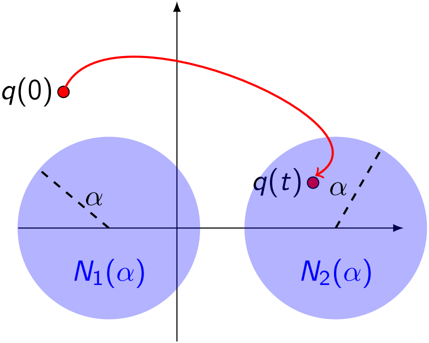
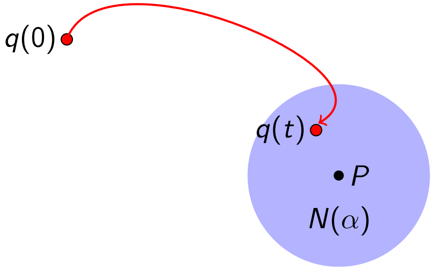


Measure of $B(\alpha)$



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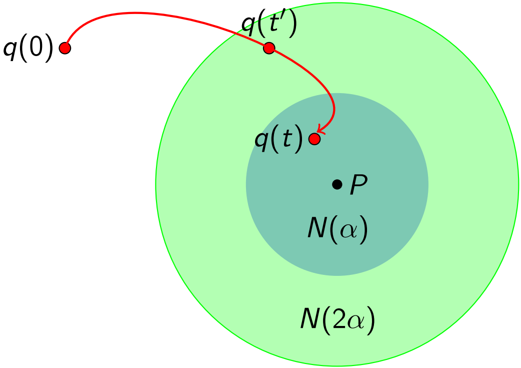

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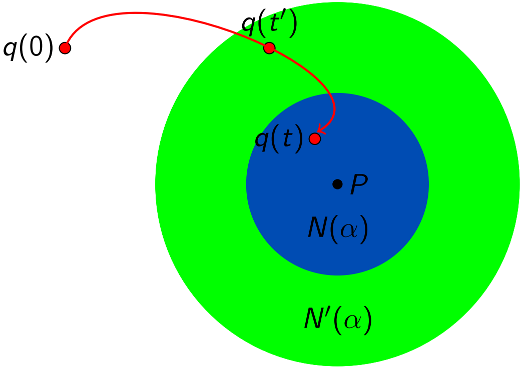

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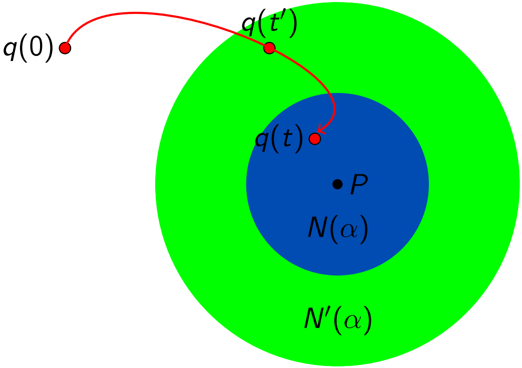

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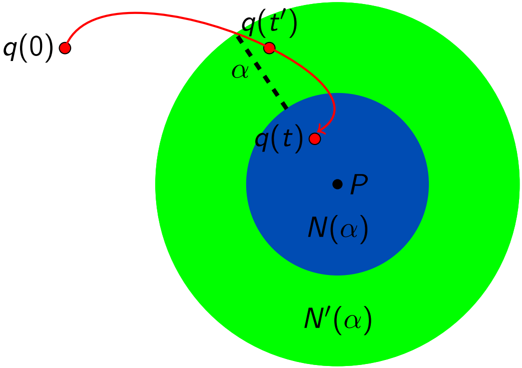

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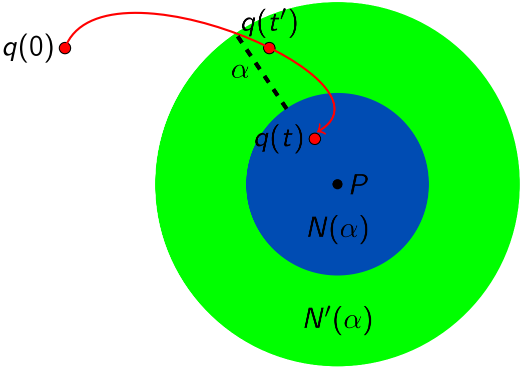

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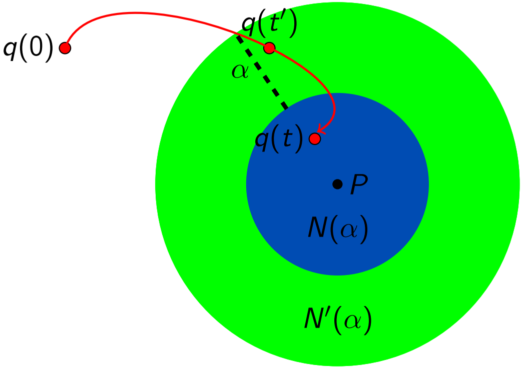

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Theorem

For the measure of initial values leading to an α-collisions in $[0, 1]$ it holds $\lambda(B(\alpha)) \leq 64\pi^2\alpha^{1.5}$.

Average case complexity

- Polynomial average time: $\int \frac{T_A(x,n)^{\varepsilon}}{n} dx$ bounded.
- $x \notin B(\alpha) \Rightarrow T(x,n) \in O((n + \frac{1}{\alpha})^m)$
- $\lambda(B(\alpha)) \leq 64\pi^2\alpha^{1.5}$

- By a similar argument as before one can show that $\lambda(A) \geq 8\pi^2$.
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Theorem

Simulating the planar circular restricted three-body problem can be done in polynomial time on average.

Conclusion / Future Work

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- How about the general N-body or other systems?